

Quantum Decision Theory*

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Version 06/15/11

Very Preliminary

Abstract

We argue that, contrary to conventional wisdom, decision theory is not invariant to the physical environment in which a decision is made. Specifically, we show that a decision maker (DM) with access to quantum information resources may be able to do strictly better than a DM with access only to classical information resources. In this respect, our findings are somewhat akin to those in computer science that have established the superiority of quantum over classical algorithms for certain problems. We treat three kinds of decision tree: (i) Kuhn trees ([24, 1950], [25, 1953]) in which the DM has perfect recall; (ii) Kuhn trees in which the DM has imperfect recall; and (iii) non-Kuhn trees.

1 Introduction

Until recently, it was widely believed that the theory of computing could be developed without attention to the particular physical components (silicon, copper, etc.) from which computers are built.¹ The conventional view of decision theory is similar: The physical manifestation of a decision problem is of no consequence to the theory.²

With the advent of quantum computing, the conventional point of view of computing has turned out to be wrong.³ The purpose of this paper is to argue that the conventional point of view in decision theory is also wrong—and for the same broad reason. We will show that the availability of **quantum information resources** can make a sharp difference to **decision making**.

2 Quantum vs. Classical Information

To set the stage, we need to begin with how classical information resources impinge on decision making. In this case, we are interested in what happens when a decision maker (DM) has access to

*We thank Samson Abramsky, Lucy Brandenburger, Elliot Lipnowski, Hong Luo, Natalya Vinokurova, Noson Yanofsky, Sasha Zolley, and a seminar audience at the CUNY Graduate Center for very helpful input. Financial support from the Stern School of Business is gratefully acknowledged. qdt-06-15-11

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¹Samson Abramsky has recounted (seminar, CUNY Graduate Center, 10/21/10) how the first axiom of computer science he learnt from his Ph.D. advisor was that, as far as software and the theory of computing are concerned, computers might as well be made of green cheese.

²Decision problems, too, could be made of green cheese.

³Prominent examples of quantum algorithms that are superior to classical algorithms are the Deutsch-Jozsa algorithm [10, 1992], Grover's algorithm [14, 1996], and Shor's algorithm [31, 1997].

classical signals and can make his choices contingent on (the realizations of) those signals. Under one interpretation, the issue is whether this extra resource of signals leads to an improvement in the DM’s position. Under a second interpretation, signals are assumed to be omnipresent in the environment, and the analysis of decision making with signals is simply the correct analysis of decision making.

Now, what is meant by the distinction between classical and quantum signals? The key difference is scale. Classical signals are encoded in the macroscopic state of some physical system—for example, in an electrical current or in pulses of light (or even in smoke signals...). Quantum signals are encoded in the microscopic state of a system—for example, in the spin of an electron or in the polarization of a photon.

The special feature of quantum signals is that they can be not only correlated but also **entangled**, where this term refers to exotic correlations which cannot arise in the classical case. Historically, the phenomenon of entanglement was seen as a conceptually troublesome aspect of quantum mechanics, but the modern view is that it is actually a distinctive and valuable resource. In the past 25 years, various information-theoretic tasks that are impossible to perform classically have been shown to be both possible and implementable using quantum resources. As one example, quantum cryptography has even reached the commercial arena (Merali [27, 2011]). The practical realization of quantum computing may not be far off.⁴

In this paper, we carry out a somewhat analogous exercise in the case of decision theory. We ask: Does giving a DM access to quantum, not just classical, signals lead to an improvement in what the DM can achieve? We will identify conditions under which this is indeed so.

A word on terminology: It is conventional in decision and game theory to talk about signals. (Toy examples of classical signaling devices are coin or die tosses.) Accordingly, we use the term “signals”—whether referring to the classical or quantum domain—in this paper. But, naturally, if we are interested in the effect of actual quantum resources on decision making, we must respect the constraints of quantum mechanics, which include the so-called **no-signaling** condition (Ghirardi, Rimini, and Weber [13, 1980]). We will review the definition of no signaling later (Section 7), when we will check that our analysis of signaling à la decision theory indeed respects this condition. Still, the terminology—signals that respect no signaling—obviously has the potential to cause confusion.

3 Decision Making

Kuhn [24, 1950], [25, 1953] introduced into game theory the crucial concept of **perfect recall**. This says that a player remembers all his previous choices and everything he previously knew. While the vast majority of work in game theory and decision theory (the concept applies unchanged to the latter) limits itself to situations of perfect recall, this is surely for reasons of simplicity. More realistically, a DM may well be subject to resource bounds, including **imperfect recall**.

Figure 1 depicts a simple example of a decision tree (as yet without payoffs) in which the DM has imperfect recall. The circular node belongs to Nature and the square nodes belong to the DM. Note that at the information set I , the DM does not remember his previous choice (if any).⁵

⁴For up-to-date information on the quest to build quantum computers, see http://www.sciencedaily.com/news/computers_math/quantum_computers/.

⁵Of course, this is not yet a formal definition of imperfect recall.

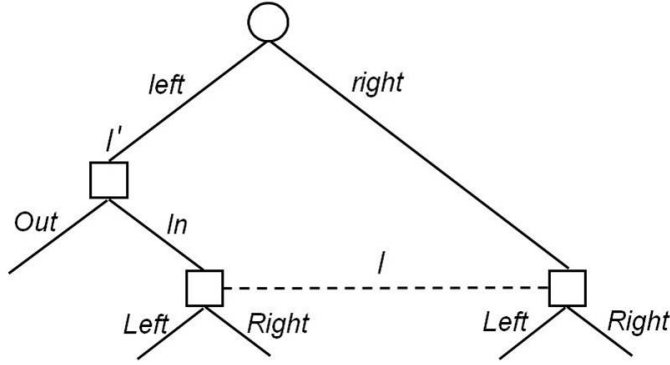


Figure 1

Intuitively, once a DM has imperfect recall, the availability of signals could help him. Signals might be able to carry information through the tree which the DM is unable to carry himself—information which might aid him in his decision making. Still, it is not immediately obvious just what types of signal might be required to bring about this benefit—nor, how the benefit might vary according to the types of memory limit present. Table 1 lays out the relationships we have discerned.

Signals					
Perfect-recall Kuhn tree	None	=	Classical	=	Quantum
Imperfect-recall Kuhn tree	None	=	Classical	<	Quantum
Non-Kuhn tree	None	<	Classical	<	Quantum

Table 1

Each row should be read from left to right, and says what happens when the type of signal indicated becomes available. An “=” (resp. “<”) sign means that the best expected payoff the DM can achieve is no larger (resp. is larger) when the extra type of signal becomes available.

The first row is a basic check of our idea. It says that when the DM has perfect recall, neither classical nor quantum signals produce any benefit. The second row says that, even when the DM has imperfect recall, classical signals produce no benefit. But, quantum signals may produce a benefit. This, then, is a place where classical and quantum theories of decision making diverge.

Next is the third row. We call a decision tree (which might have perfect or imperfect recall) a **Kuhn tree** if every path from the root of the tree to a terminal node passes through each information set at most once; otherwise, it is a **non-Kuhn tree**. (Kuhn’s definition of a tree includes this condition—hence our terminology.) Intuitively, a non-Kuhn tree indicates that the DM exhibits a more ‘serious’ type of memory limitation as compared with even an imperfect-recall Kuhn tree. Non-Kuhn trees were first studied by Isbell [21, 1957]. Figure 2 depicts (again, as yet without payoffs) the well-known example of such a tree—called the Absent-Minded Driver’s Problem—due to Piccione and Rubinstein [29, 1997]. The third row of Table 1 says that, unlike in Kuhn trees, a DM

may benefit even from access to classical signals. He may benefit further from access to quantum signals.

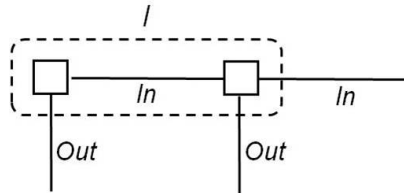


Figure 2

4 Signals

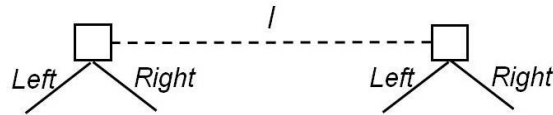


Figure 3

We now give a little more detail on what we mean by giving a DM access to **signals**. We will give two formulations of this idea. Under the first, we assume that, at each of his information sets, the DM may observe the value of a random variable and is allowed to make his choice contingent on the realization of the r.v.. The r.v.'s may be correlated (i.e., dependent) across information sets. Figure 3 shows an information set I lying in some decision tree, and Figure 4 shows what I would look like in the tree augmented with signals of this first kind. (Here, the signal is a simple coin toss at I , on which the DM can peg his choice.)

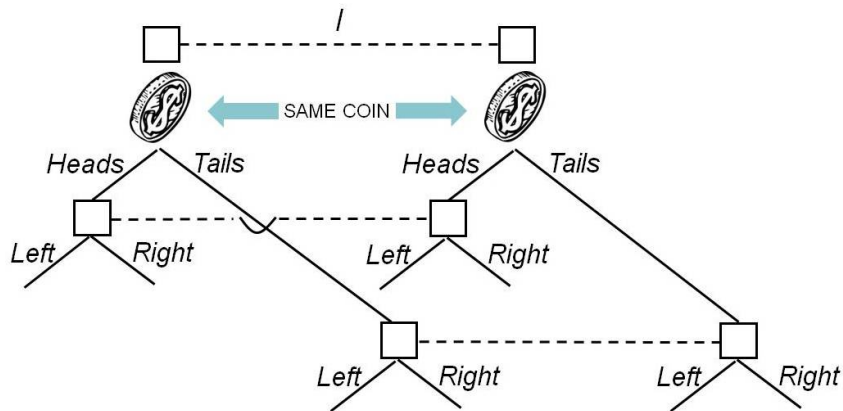


Figure 4

Our second formulation assumes that, even within a given information set, at each node the DM may observe the value of a different r.v.. But, we do require that the different r.v.'s at the given

information set are **exchangeable**. (In particular, they must have the same range.) Figure 5 is then the analog to Figure 4. Of course, this second formulation is a generalization of the first, since it includes the case where there is one common r.v.. The second formulation seems to us to be conceptually more appropriate. The idea of an information set is that the DM cannot distinguish among the nodes in it—not that the nodes are literally the same. So, when we add signals, it seems right to allow different signals at different nodes, provided that the signals are indistinguishable. This is ensured by exchangeability.⁶

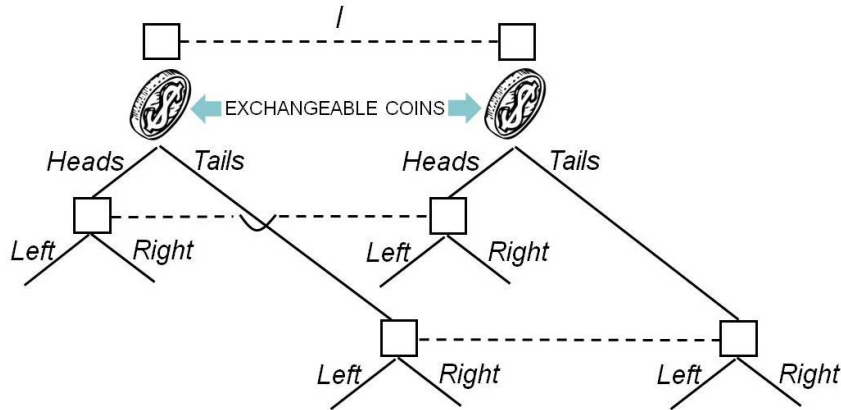


Figure 5

It will turn out that the first two rows of Table 1 hold under both formulations of signals. However, the third row uses the broader formulation.

5 Interpretation

One interpretation we can give to our analysis is that it says what happens when a DM literally sets up or engineers quantum signals. He does this because, if he is subject to some kind of memory limit, such signals can (at least, partially) make up for this deficit. The technology to use quantum information resources in this way now exists.

A second interpretation is that signals—including quantum signals—are already present in the environment. Indeed, they may actually be a constituent of memory. This is a highly speculative interpretation, because the issue of whether or not quantum processes are involved in the operation of the brain is a contentious one. (Hameroff and Penrose [15, 1995] have advanced an argument in favor. For a skeptical review, see Koch and Hepp [23, 2006].) Of course, our paper does not add to the actual neuroscience of this issue, but, it does show how, in principle, entanglement creates a memory-like effect.

There is a growing literature on quantum games. (See Meyer [28, 1999], Eisert, Wilkens, and Lewenstein [11, 1999], Eisert and Wilkens [12, 2000], Benjamin and Hayden [4, 2001], Huberman and Hogg [19, 2003], Cleve et al. [8, 2004], Iqbal and Weigert [20, 2004], Dahl and Landsburg [9, 2005],

⁶If two decision nodes in the same information set are physically separated, it seems especially appropriate to allow different signals at each. Indeed, the two nodes might be so distant in space-time that it would be impossible for a single signal to reach both. (In the relativistic setting, the nodes might have no common past.)

La Mura [26, 2005], Kargin [22, 2008], and Brandenburger [6, 2010], among others.) Our paper is concerned with what could be said to be the logically prior case of quantum decision theory.⁷

Cabello and Calsamiglia [7, 2005] also investigate quantum decision theory. They study the effect of quantum signals on the Absent-Minded Driver’s Problem ([29, 1997]). Our results on this model are in Section 8.

Hillas, Kohlberg, and Pratt [18, 2007] and Stein, Parrilo, and Ozdaglar [32, 2010] are other papers connecting exchangeability and classical correlation (in games not decision problems).

6 Classical Baseline

We now formulate and prove our results. Our treatment will remain at a somewhat heuristic level. The precise definitions of Kuhn vs. non-Kuhn tree, perfect vs. imperfect recall, etc., require some set-up. We give fully formal definitions in the Appendix.

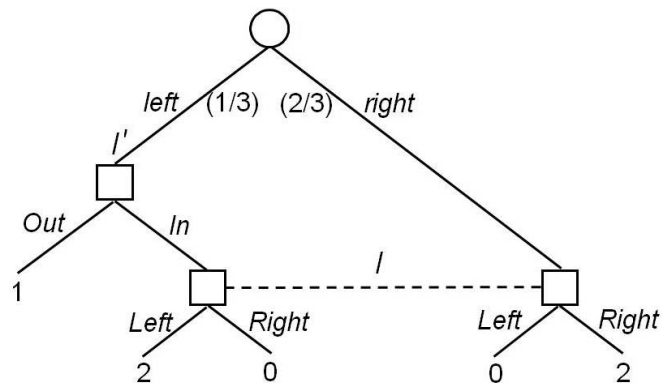


Figure 6

Figure 6 is Figure 1 repeated, this time with probabilities on moves by Nature (in parentheses) and payoffs added. Table 2 is the associated decision matrix, giving the expected payoff associated with each strategy. We see that the DM’s highest expected payoff is $5/3$.

<i>In-Left</i>	$2/3$
<i>In-Right</i>	$4/3$
<i>Out-Left</i>	$1/3$
<i>Out-Right</i>	$5/3$

Table 2

Now add a signal at I' and another signal at I . Figure 7 gives the idea in a simplified case, in which we have added a coin toss at I' and another—possibly correlated—coin toss at I . Two observations on Figure 7: First, the information set corresponding to I in the underlying tree contains

⁷Much classical decision theory came after the beginnings of classical game theory, and, likewise, can be viewed as logically prior. Thus, Savage [30, 1954], Anscombe and Aumann [2, 1963], and all that followed came after von Neumann and Morgenstern [33, 1944].

an additional node. There are now two nodes after the DM chooses In , not one node as before. This reflects an assumption that the DM does not remember the signal ($Heads'$ vs. $Tails'$) that he received at I' . (Making the opposite assumption would seem to be changing the memory constraints we have imposed on the DM.) Second, we emphasize that, in accordance with the usual reading of a tree diagram, the three coin images at this information set depict a single physical coin.

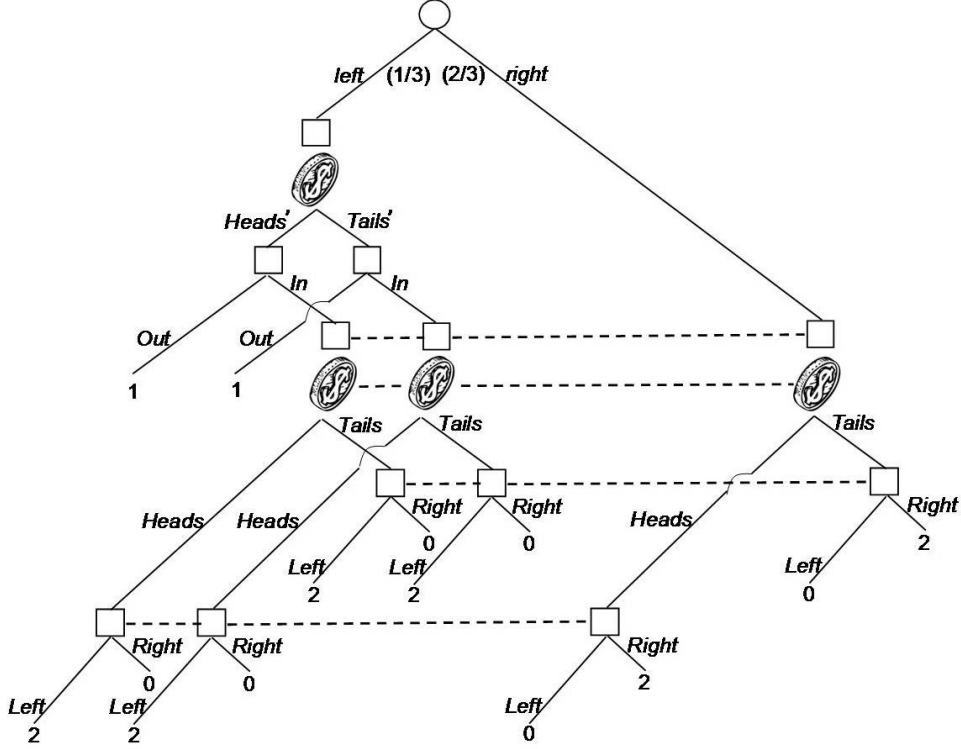


Figure 7

In general, given information sets I, I', \dots , in a tree, let $M_I, M_{I'}, \dots$, be the associated sets of moves and let $\Omega_I, \Omega_{I'}, \dots$, be associated signal spaces that are added to the underlying tree. (If there is no signal at an information set, the signal space is a singleton.) Let $M = M_I \times M_{I'} \times \dots$ and $\Omega = \Omega_I \times \Omega_{I'} \times \dots$. Fix a probability measure μ on Ω . A strategy for the DM in the underlying tree is simply an element $m \in M$. A strategy for the DM in the extended tree is a sequence of maps $f_I : \Omega_I \rightarrow M_I, f_{I'} : \Omega_{I'} \rightarrow M_{I'}, \dots$. Let $f = f_I \times f_{I'} \times \dots$. Finally, write $\pi(m)$ for the expected payoff to the DM in the underlying tree, when he chooses strategy m and we average over Nature. Then, the expected payoff to the DM in the extended tree, when he chooses strategy f (and we again average over Nature), is $\sum_{m \in M} (\mu \circ f^{-1})(m) \times \pi(m)$. That is, in the tree with signals, the expected payoff to any particular strategy is a convex combination of expected payoffs to strategies in the underlying tree. It follows that:

Proposition 6.1 *The highest expected payoff to a DM in a Kuhn tree with classical signals is the same as that in the tree without signals.*

There is an important assumption built into this proposition, as even the simplest examples show. Consider the tree depicted in Figure 8, and the joint distribution of Nature and a coin toss at the DM's information set depicted in Figure 9. The DM gets an expected payoff of 1 (in fact, 1 almost surely) by choosing *Left* after *Heads* and *Right* after *Tails*. His (highest) expected payoff in the underlying tree is $1/2$. The reason for the difference is, of course, that we have allowed correlation between the signal and Nature. Independence of signals and Nature was built into our argument above when we first averaged over Nature and then averaged over signals. This seems like the right assumption for our purpose, which is to analyze signals as memory substitutes (or even memory itself). We do not want signals to allow the DM to obtain information he never had.

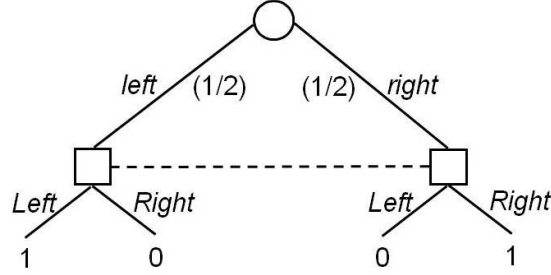


Figure 8

	<i>left</i>	<i>right</i>
<i>Heads</i>	1/2	0
<i>Tails</i>	0	1/2

Figure 9

7 Quantum Improvement

We now show:

Proposition 7.1 *There is a Kuhn tree (with imperfect recall) in which the DM can achieve a higher expected payoff with quantum signals than with any classical signals.*

The decision tree of Figure 10 will serve to establish this claim. It is assumed that $0 < m < M$. First, we claim that the DM's expected payoff with classical signals is at most 0. To see this, start without signals. Observe that, in Figure 10, the only way for the DM to get the $+m$ payoff (at least, with positive probability) is if he chooses u at *East* and t at *South*. But then, to avoid the $-M$ payoff on the right side of the tree, he must choose B at *North*. But then, to avoid the upper $-M$ payoff on the left side, he must choose U at *West*. But then, to avoid the lower $-M$ payoff on the left side, he must choose b at *South*, not t as we supposed. It follows that the $+m$ payoff cannot arise unless at least one $-M$ payoff also arises (with the same probability). Since $M > m$, we have now shown that the DM's expected payoff in Figure 10 is at most 0. By Proposition 6.1, this is also the DM's highest possible expected payoff in any extended tree with signals.

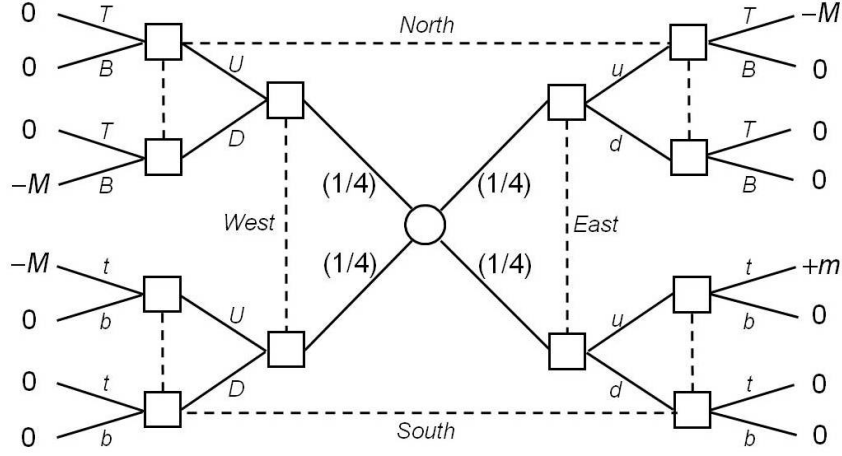


Figure 10

Next, consider the signals given by Table 3 (where $\varphi = \frac{2}{\sqrt{5}+1}$ is the inverse of the golden ratio):

		GG	RG	GR	RR
West	North	φ^3	φ^2	φ^2	0
East	North	0	φ	φ^3	φ^4
West	South	0	φ^3	φ	φ^4
East	South	φ^5	φ^4	φ^4	φ

Table 3

At each of the four information sets, there is a signal which can take the value G (for *Green*) or R (for *Red*). If the path through the tree hits the information set *West* followed by the information set *North*, then the associated string of signals is GG with probability φ^3 , RG with probability φ^2 , GR with probability φ^2 , and RR with probability 0. Likewise, for *East* followed by *North*, for *West* followed by *South*, and for *East* followed by *South*, the probabilities of each of the four possible strings of signals are given in the corresponding row. Suppose, the DM responds to signals as follows. At *West*, he selects $G \rightarrow U$ and $R \rightarrow D$. At *East*, $G \rightarrow u$ and $R \rightarrow d$. At *North*, $G \rightarrow T$ and $R \rightarrow B$. At *South*, $G \rightarrow t$ and $R \rightarrow b$. Then, we see that the DM achieves an expected payoff of $1/4 \times \varphi^5 \times m > 0$.

The signals in Table 3 can be generated quantum mechanically. We will not describe the physics in this paper, but refer the reader to Hardy [16, 1992], [17, 1993], where the physical system yielding Table 3 is described. (This is the system that Hardy uses to establish his strengthening of Bell's famous no-go theorem ([3, 1964].) By Proposition 6.1, we know that these signals cannot be generated in the classical domain. Table 4 gives one way of seeing why.

	<i>West</i>	<i>North</i>	<i>East</i>	<i>South</i>
$\omega^{(1)}$	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>
$\omega^{(2)}$	<i>R</i>	<i>G</i>	<i>G</i>	<i>G</i>
$\omega^{(3)}$	<i>G</i>	<i>R</i>	<i>G</i>	<i>G</i>
$\omega^{(4)}$	<i>G</i>	<i>G</i>	<i>R</i>	<i>G</i>
$\omega^{(5)}$	<i>G</i>	<i>G</i>	<i>G</i>	<i>R</i>
$\omega^{(6)}$	<i>R</i>	<i>R</i>	<i>G</i>	<i>G</i>
$\omega^{(7)}$	<i>G</i>	<i>R</i>	<i>R</i>	<i>G</i>
$\omega^{(8)}$	<i>G</i>	<i>G</i>	<i>R</i>	<i>R</i>
$\omega^{(9)}$	<i>R</i>	<i>G</i>	<i>G</i>	<i>R</i>
$\omega^{(10)}$	<i>R</i>	<i>G</i>	<i>R</i>	<i>G</i>
$\omega^{(11)}$	<i>G</i>	<i>R</i>	<i>G</i>	<i>R</i>
$\omega^{(12)}$	<i>R</i>	<i>R</i>	<i>R</i>	<i>G</i>
$\omega^{(13)}$	<i>G</i>	<i>R</i>	<i>R</i>	<i>R</i>
$\omega^{(14)}$	<i>R</i>	<i>G</i>	<i>R</i>	<i>R</i>
$\omega^{(15)}$	<i>R</i>	<i>R</i>	<i>G</i>	<i>R</i>
$\omega^{(16)}$	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>

Table 4

The rows are the points in the signal space $\Omega = \{G, R\}^4$. Let μ be the probability measure on Ω . We need μ to marginalize to the probability measures in Table 3. The 0 in the first row of Table 3 then imposes the requirement:

$$\mu(\omega^{(6)}) + \mu(\omega^{(12)}) + \mu(\omega^{(15)}) + \mu(\omega^{(16)}) = 0.$$

Likewise, the 0's in the second and third rows of Table 3 impose the requirements:

$$\begin{aligned} \mu(\omega^{(1)}) + \mu(\omega^{(2)}) + \mu(\omega^{(5)}) + \mu(\omega^{(9)}) &= 0, \\ \mu(\omega^{(1)}) + \mu(\omega^{(3)}) + \mu(\omega^{(4)}) + \mu(\omega^{(7)}) &= 0. \end{aligned}$$

From these requirements we conclude that $\mu(\omega^{(i)}) = 0$ for $i = 1, 2, 3, 4, 5, 6, 7, 9, 12, 15, 16$. But, from the fourth row of Table 3 we get the requirement:

$$\mu(\omega^{(1)}) + \mu(\omega^{(2)}) + \mu(\omega^{(3)}) + \mu(\omega^{(6)}) = \varphi^5 > 0,$$

which is then impossible.

In the terminology of Abramsky and Brandenburger [1, 2011], the probabilities in Table 3 are not **extendable** to probabilities on a joint signal space. When we talk about signals that arise in the classical domain, we mean precisely signals that are generated by a joint signal space. Extendability holds by definition. But, a hallmark of the quantum domain is that signals such as those in Table 3 can arise, which are not extendable.⁸ The underlying reason is that the signals at *West* and *East*

⁸The Bell and Hardy theorems are usually described as establishing the impossibility of a local hidden-variable analysis of quantum mechanics. Abramsky and Brandenburger [1, 2011] show that there is a **factorizable** hidden-variable analysis if and only if extendability holds. Factorizability specializes to locality in the Bell and Hardy set-ups.

arise from incompatible measurements on a certain particle (a photon, say), while the signals at *North* and *South* arises from incompatible measurements on a second particle. Each of the four strings of signals in Table 3 (*GG*, *RG*, *GR*, *RR*) comes from one measurement on the first particle and one measurement on the second particle, and is therefore physically possible. (The particular probabilities reflect the fact that the two particles are entangled.) But, there can be no joint signal space, because this would involve incompatible measurements.

We said in the Introduction that features of quantum mechanics that were first seen as conceptually troublesome are now understood to be valuable resources. Incompatibility is one such feature. Prima facie, incompatibility is a new and curious constraint on what measurements can be made on a physical system. In fact, we see that it lifts a constraint because, now, signals are possible that do not arise from a joint signal space.

The probabilities in Table 3 are quantum mechanical, and so obey the **no-signaling** condition (Ghirardi, Rimini, and Weber [13, 1980]). To take an example, fix *West* and *North*. The marginal probability of *G* at *West* is $\varphi^3 + \varphi^2$. Next, fix *West* and *South*. The marginal probability of *G* at *West* is $0 + \varphi$. But, $\varphi^3 + \varphi^2 = \varphi$, so these two probabilities are equal. (It can be checked that the other analogous equalities all hold under the probabilities in Table 3.) At the physical level, equality says that the probability of a particular outcome of a measurement made on one particle does not depend on the measurement made on another—perhaps, spatially separated—particle. Satisfaction of this condition is usually said to be necessary if a physical theory is to be compatible with relativity. The conclusion for us (which is a bit confusing as far as terminology goes) is that signals in the decision-theoretic sense, while satisfying no-signaling in the physical sense, can nevertheless have a significant effect on decision making.

Finally, in this section, we turn to the case of Kuhn trees with perfect recall. Fix such a tree and some information set *I* in the tree. At *I*, the DM knows all of his previous choices and everything that he previously knew about Nature’s choices (and, perhaps, now knows more about Nature’s choices). These are formal statements; see Lemmas A.6 and A.7 in the Appendix. It follows that the only effect of a signal at *I* can be to add to the DM’s information at *I* about Nature’s choices. But, this is possible only if the signal is correlated with Nature, which we have ruled out. This gives us the first row of Table 1. In fact, this argument holds regardless of whether the signals are classical, quantum, or even superquantum:

Proposition 7.2 *The highest expected payoff to a DM in a perfect-recall Kuhn tree with (even superquantum) signals is the same as that in the tree without signals.*

8 Exchangeable Signals

We next look at our second formulation of signals, which allows for a different—but exchangeable—signal at each node within a given information set. This formulation leaves unchanged all our results so far. The reason is that in a Kuhn tree (whether with perfect or imperfect recall), every path from the root to a terminal node passes through a given information set at most once. But, the exchangeability formulation can make a big difference in non-Kuhn trees. Figure 11 is Figure 2 repeated, this time with payoffs added. (These are the payoffs for the Piccione-Rubinstein [29, 1997] Absent-Minded Driver’s Problem.) The highest (expected) payoff the DM can achieve without signals is 1, obtained by choosing *In* (vs. 0 from choosing *Out*).

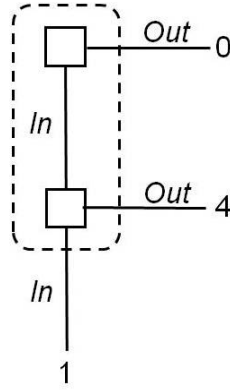


Figure 11

Now add classical signals. Specifically, we add one coin toss (*Heads* vs. *Tails*) at the first node and a second coin toss (*Heads'* vs. *Tails'*) at the second node. The two coins are physically indistinguishable. Figure 12 depicts the extended tree, and Figures 13 and 14 two possible joint distributions of the coin tosses. In Figure 12, the DM has two information sets, depending on whether he sees a coin land *Heads/Heads'* or *Tails/Tails'*. The three nodes in the first information set are shaded with the right-side-up triangles, and the three nodes in the second information set with upside-down triangles.

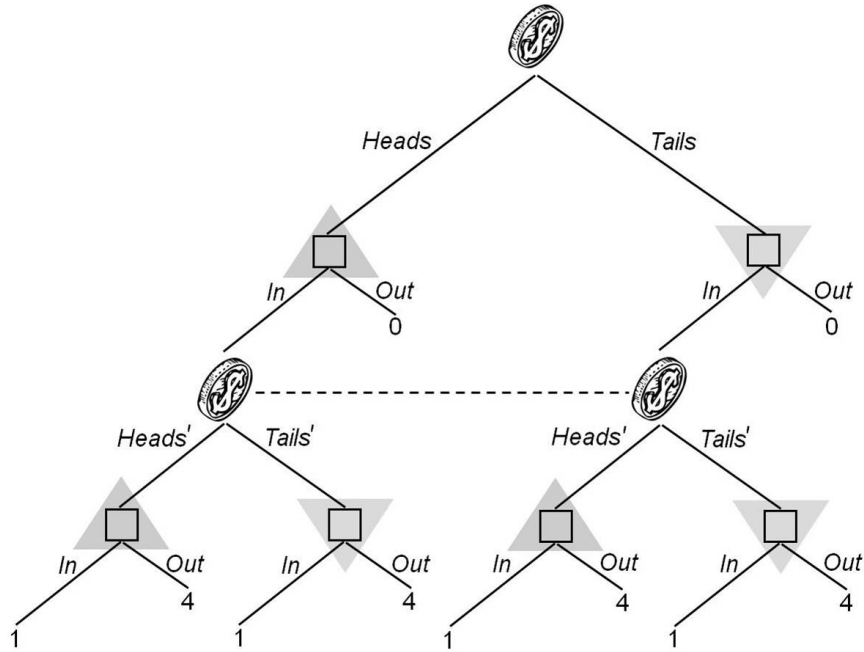


Figure 12

	<i>Heads'</i>	<i>Tails'</i>
<i>Heads</i>	1/4	1/4
<i>Tails</i>	1/4	1/4

Figure 13

	<i>Heads'</i>	<i>Tails'</i>
<i>Heads</i>	0	1/2
<i>Tails</i>	1/2	0

Figure 14

Suppose the DM chooses *In* after observing *Heads/Heads'*, and *Out* after observing *Tails/Tails'*. His expected payoff with the signals in Figure 13 is then $\frac{1}{2} \times 0 + \frac{1}{4} \times 4 + \frac{1}{4} \times 1 = 5/4 > 1$, where 1 was the best achievable without signals. Of course, the distribution in Figure 13 is exchangeable, since it is i.i.d.. (The use of i.i.d. signals in non-Kuhn trees goes back to Isbell [21, 1957].) In fact, the DM can do even better with the distribution in Figure 14, which is again exchangeable, though no longer i.i.d.. His expected payoff is now $\frac{1}{2} \times 0 + \frac{1}{2} \times 4 = 2 > 5/4$.

We see that in a non-Kuhn tree, there can be improvement even with classical signals. Proposition 6.1 (which immediately extends to exchangeable signals) says that this cannot happen in a Kuhn tree. As a next step, we could glue together a simple non-Kuhn tree such as Figure 11 with a tree such as Figure 10, to obtain another non-Kuhn tree, on which we could repeat the argument leading to Proposition 7.1. This would show that quantum signals can improve still further on classical signals in non-Kuhn trees, as the third row of Table 1 indicates.

Cabello and Calsamiglia [7, 2005] take a somewhat different approach to the tree of Figure 11. They create the distribution of Figure 14 via quantum resources rather than via classical resources. Their argument is that classical systems—here, there is a system at each node, consisting of a coin and a coin-tossing device—are ultimately distinguishable. The DM could then distinguish between the two nodes by distinguishing between the two systems. This violates the ‘rules of the game’ of the original set-up. Cabello and Calsamiglia prefer to get the anti-correlation in Figure 14 via a pair of qubits, on the argument that indistinguishability is then guaranteed from basic principles (the quantum state is a complete description).

We are not fully convinced that the use of classical signals to create the distribution of Figure 14 is invalid. The precise requirement is that the observations the DM makes at the two nodes do not allow him to distinguish the nodes. Perhaps, the DM observes only the outcomes of the coin tosses, and not the details of the coins and coin-tossing devices. If so, it is not obvious that he will be able to distinguish the nodes. We prefer to argue for an effect of classical signals in non-Kuhn trees, and for an additional effect of quantum signals.

9 Conclusion

This paper is about bringing quantum mechanics to bear on decision theory. What about bringing decision theory to bear on quantum mechanics? Perhaps, our paper can bring a new perspective to the enterprise of trying to characterize quantum mechanics on the basis of information-theoretic principles. We hope to be able to report on this direction in future work.

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A Formal Definition of a Decision Tree

A decision tree is a two-person game in extensive form, where one player is the decision maker (DM) and the other is Nature. We now give formal definitions following Kuhn [24, 1950], [25, 1953]. The presentation follows that in Brandenburger [5, 2007].

Definition A.1 A (*finite*) *decision tree* consists of:

- (a) A set of two players, one called the **decision maker (DM)** and the other called **Nature**.
- (b) A finite rooted tree.
- (c) A partition of the set of non-terminal nodes of the tree into two subsets denoted N (with typical element n) and M (with typical element m). The members of N are called **decision nodes**, and the members of M are called **chance nodes**.
- (d) A partition of N (resp. M) into **information sets** denoted I (resp. J) such that for each I (resp. J):
 - (i) all nodes in I (resp. J) have the same number of outgoing branches, and there is a given 1-1 correspondence between the sets of outgoing branches of different nodes in I (resp. J);
 - (ii) every path in the tree from the root to a terminal node crosses each I (resp. J) at most once.

For each information set I (resp. J), number the branches going out of each node in I (resp. J) from 1 through $\#I$ (resp. $\#J$) so that the 1-1 correspondence in (d.i) above is preserved.

Definition A.2 A **strategy** (for the DM) associates with each information set I , an integer between 1 and $\#I$, to be called the DM’s **choice** at I . Let S denote the set of strategies for the DM. A **state of the world** (or **state**) associates with each information set J , an integer between 1 and $\#J$ to be called the **choice** of Nature at J . Let Ω denote the set of states.

Note that a pair (s, ω) in $S \times \Omega$ induces a unique path through the tree.

Definition A.3 Fix a node n in N and a strategy s . Say n is **allowed under** s if there is a state ω such that the path induced by (s, ω) passes through n . Say an information set I is **allowed under** s if some n in I is allowed under s .

Definition A.4 Say the DM has **perfect recall** if for any strategy s , information set I , and nodes n and n^* in I , node n is allowed under s if and only if node n^* is allowed under s .

Definition A.5 Say a node n in N is **non-trivial** if it has at least two outgoing branches.

Define a relation of precedence on the DM's information sets I , as follows: Given two information sets I and I' , say that I **precedes** I' if there are nodes n in I and n' in I' such that the path from the root to n' passes through n . It is well known that if the DM has perfect recall and all decision nodes are non-trivial, then this relation is irreflexive and transitive, and each information set I has at most one immediate predecessor. (Proofs of these assertions can be found in Brandenburger [5, 2007, Appendix], or can be constructed from arguments in Wilson [34, 1972].)

Kuhn [25, 1953, p.213] observes that perfect recall implies that the DM remembers: (i) all of his choices at previous nodes; and (ii) everything he knew at those nodes. The following two lemmas formalize these observations. (The proofs are in [5, 2007, Appendix].)

Lemma A.6 *Suppose the DM has perfect recall and all decision nodes are non-trivial. Fix information sets I and I' , and strategies s and s' . Suppose that I' is allowed under both s and s' , and I precedes I' . Then I is allowed under both s and s' , and s and s' coincide at I .*

Next, write:

$$[I] = \{\omega : I \text{ is allowed under } \omega\}.$$

Lemma A.7 *Suppose the DM has perfect recall and all decision nodes are non-trivial. Fix information sets I and I' . If I' succeeds I , then $[I'] \subseteq [I]$.*